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## Fermionic zero-modes around string solitons

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### Abstract

*In presence of string solitons, index theorems for the generalised Dirac operators have to be revisited. We show that in supersymmetric configurations the fermionic operators decouple, so that there are no mixing effects between different fermions in the index theorems. We extend the index theorems in presence of torsion to the generic case of manifolds with boundary, which naturally appear in string solutions and apply this result to the soliton solution by Callan, Harvey and Strominger.*

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The existence of solitons means that a full non-perturbative theory may have a much richer structure than it is apparent in perturbation theory: since it is commonly believed that pointlike quantum field theories are not adequate to unify all the four fundamental forces, including gravity, and that they should be substituted by string theory [1], it is important to study soliton solutions in this context. In the last few years much attention has been paid to the existence of these solutions after the paper by Strominger [2], who showed that supergravity in ten dimensions coupled to super-Yang-Mills, which is the field theory limit of the heterotic string, has as solitonic solution the heterotic fivebrane (a five-dimensional extended object), already conjectured by Duff two years before [3]. This fivebrane is everywhere nonsingular and carries a topological charge, just as usual 't Hooft instantons in four dimensions. After this solution many others were found [4], some of which were exact to all orders in  $\alpha'$  the inverse string tension and which, in certain limits, corresponded to exact conformal field theories given by a Wess-Zumino-Witten model times a one-dimensional Feigin-Fuchs Coulomb gas [5]. The common feature is given by the presence of a Yang-Mills instanton in the four directions transverse to the fivebrane. Just as in the 4-dimensional instantonic case, we are interested in the number and structure of the fermionic zero-modes, whose presence could drive the breaking of chiral symmetries and/or supersymmetry, through the formation of gaugino [6, 7] or gravitino condensates [8, 9, 10]. A first step therefore is to apply the index theorems to the fermionic kinetic operators to compute the number of zero-modes these solitonic solutions have. A novel feature is the presence of the 3-form  $H$  (the field-strength of the two form  $B$  plus Chern-Simons terms) in the supergravity Lagrangian, which plays the role of a torsion and which may affect the index formulae. The importance played by the antisymmetric tensor in superstring theory and in compactification has been greatly stressed by Rohm and Witten [11] and by Strominger [12]; up to now the Dirac index formula for manifolds with torsion [13]

has been known only in the case of compact manifolds without boundary and studied in a totally different context. In our case we want to get in touch with the kind of Dirac operators arising from the compactification of superstring inspired supergravity Lagrangians. We propose an extension of this formula to the case of manifolds with boundary and apply it to the solution of Callan, Harvey and Strominger (CHS) [5] and derive the exact form of the fermionic operators; from there we will infer the number of zero-modes for gaugino and dilatino. The formulae we derive have a wide applicability, since we have found that for supersymmetric solutions there is a decoupling among the fermionic operators which makes possible to neglect mixing effects between the different fermion indices, although singularities in the string solution may require a subtle extension. In Section 1 we briefly sketch the conditions for having supersymmetric backgrounds exact to all orders in  $\alpha'$ . In Section 2 we explicitly calculate the fermionic equations of motion in supersymmetric backgrounds and show their factorisation. In Section 3 we review the formulation of Dirac index theorem in presence of torsion and introduce our generalisation to manifolds with boundaries. Finally in Section 4 we apply the results of Sect. 3 to a specific soliton solutions and calculate the number of zero-modes of the fermionic operators.

## 1 Supersymmetric backgrounds

First of all, let us introduce the model we want to study: we start from the  $D = 10$   $N = 1$  supergravity and super Yang-Mills action [14] which coincides with the leading terms coming from the heterotic string theory. The bosonic part of the action is:

$$S_B = -\frac{1}{2} \int d^{10}x \sqrt{g} e^{-2\Phi} \left( R - 4(\nabla\Phi)^2 + \frac{1}{3} H_{MNP} H^{MNP} + \frac{\alpha'}{30} \text{Tr} F_{MN} F^{MN} \right), \quad (1)$$

where  $H$  is related to the antisymmetric tensor  $B$  by the following relation:

$$H = dB + \alpha' \left( \Sigma_3^L(\Omega_-) - \frac{1}{30} \Sigma_3^{YM}(A) \right), \quad (2)$$

where  $\Sigma_3$  stands for the Chern-Simons three-form, so that we obtain modified Bianchi identities for  $H$ :

$$dH = \alpha' \left( \text{tr } R(\Omega_-) \wedge R(\Omega_-) - \frac{1}{30} \text{Tr } F \wedge F \right). \quad (3)$$

The trace  $\text{Tr}$  is over the adjoint representation of the gauge group, and the curvature  $R$  is built with  $\Omega_{-M}{}^{AB} = (\omega_M{}^{AB} - H_M{}^{AB})$ , a generalised spin connection with torsion. Instead of solving the equations of motions for this action, it is more convenient and rewarding to look for bosonic backgrounds annihilated by some of the  $N = 1$  supersymmetry transformations, since only the vacuum is annihilated by all of them. The supersymmetry transformations of the fermionic fields are:

$$\begin{aligned} \delta\chi &= -\frac{1}{4} F_{MN} \Gamma^{MN} \varepsilon, \\ \delta\lambda &= -\frac{1}{4} \left( \Gamma^M \partial_M \Phi - \frac{1}{6} H_{MNP} \Gamma^{MNP} \right) \varepsilon, \\ \delta\psi_M &= \left( \partial_M + \frac{1}{4} \Omega_{-M}{}^{AB} \Gamma_{AB} \right) \varepsilon. \end{aligned} \quad (4)$$

For simplicity we take all fields independent on the six Minkowskian coordinates of the fivebrane. In addition we must take into account that the supersymmetry spinor in ten dimensions belongs to the representation **16** of  $SO(1, 9)$  and that under  $SO(1, 5) \times SO(4)$  it breaks in  $\mathbf{16} = (\mathbf{4}, \mathbf{2}_+) \oplus (\mathbf{4}^*, \mathbf{2}_-)$  which we call  $\varepsilon_{\pm}$ . The  $D = 4$  bosonic background we are interested in annihilates  $\varepsilon_+$ : then we find that the most general solution is given by the configuration:

$$F = \pm * F, \quad H = \pm * d\Phi, \quad R(\Omega) = \pm * R(\Omega), \quad (5)$$

where  $\Omega = (\omega + H)$  stands for a generalised spin connection with torsion. If in addition we require a solution exact to all orders in  $\alpha'$ , it is possible to show that we are forced to identify the generalised spin connection with the gauge field through the procedure known as standard embedding. It is then possible to distinguish two

cases; either the dilaton and the three-form  $H$  are taken to be zero and the metric is self-dual, corresponding to a gravitational instanton[15, 16], or the dilaton and  $H$  are non-trivial while the metric is conformally flat [17, 19, 5] leaving us with a  $SU(2)$  selfdual connection  $\Omega_{-M}^{AB}$ , while  $\Omega_{+M}^{AB}$  is anti-selfdual. To make the gaugino variation vanish it suffices to take the gauge field to be a 't Hooft instanton;  $\delta\chi$  vanishes if  $\varepsilon = (\mathbf{4}, \mathbf{2}_+)$  and  $F_{\mu\nu} = \tilde{F}_{\mu\nu}$ , where Greek indices run from 1 to 4, the coordinates transverse to the fivebrane, We take the field strength of the antisymmetric tensor to be  $H_{\mu\nu\rho} = -\sqrt{g}\varepsilon_{\mu\nu\rho}{}^\sigma\partial_\sigma\Phi$  to annihilate the negative chirality dilatino  $\lambda$  transformation and the four dimensional metric conformally flat  $g_{\mu\nu} = e^{2\Phi}\delta_{\mu\nu}$ . Taking a constant positive chirality supersymmetry spinor we also annihilate the positive chirality gravitino variation; finally we make the identification  $R_{\mu\nu}(\Omega_+)^{ab} = \frac{1}{2}\bar{\eta}^{I,ab}F_{\mu\nu}^I$ , where  $\bar{\eta}_{ab}^I$  are the 't Hooft symbols, in order to annihilate the  $\alpha'$  correction to the Bianchi identities and, more in general, all the other corrections in  $\alpha'$ . As already alluded to, this procedure has a close analogy with the standard embedding in Calabi-Yau compactification of the heterotic string [22, 12] used to get an  $N = 1$   $D = 4$  supersymmetric background from an  $N = 1$   $D = 10$  string theory<sup>2</sup>. In string soliton solutions, the embedding is between an  $SU(2)$  connection with torsion of the four dimensional manifold transverse to the fivebrane and an  $SU(2)$  subgroup of the second  $E_8$ . The Bianchi identities become  $e^{-2\Phi}\square e^{2\Phi} = 0$ , where the d'Alembertian is intended to be evaluated in flat Euclidean space. According to the number of coordinates  $\Phi$  is supposed to depend on, one has

1. five-branes, characterised by a dilaton field  $e^{2\Phi} = e^{2\Phi_0} + \sum_i \frac{Q_i}{(x-x_i)^2}$ , considered by CHS [5];

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<sup>2</sup>In that case the embedding is between the  $SU(3)$  spin connection corresponding to the internal manifold holonomy group and the connection of an  $SU(3)$  subgroup of the gauge group, that is  $SO(32)$  or  $E_8 \times E_8$ . In the heterotic string case this embedding is in the first  $E_8$  corresponding to the gauge group which will give rise to the low energy phenomenology, while the second  $E_8$  corresponds to the so-called *hidden sector* which interacts only gravitationally with the particles belonging to the first sector.

2. monopoles, with  $e^{2\Phi} = e^{2\Phi_0} + \sum_i \frac{m_i}{|\vec{x} - \vec{x}_i|}$  found in [18, 19, 20]; this solution corresponds in four dimensional Minkowskian space to pointlike topological defects localised in the points  $\vec{x}_i$ ;
3. strings with  $e^{2\Phi} = e^{2\Phi_0} + \sum_i q_i \log |z - z_i|^2$ , where  $z = x_1 + ix_2$ , corresponds to one-dimensional defects [21];
4. domain walls with  $e^{2\Phi} = e^{2\Phi_0} + \sum_i c_i(t - t_i)$ , corresponding to two-dimensional topological defects [21].

Under Buscher's duality [32] these solutions are related to purely gravitational backgrounds with zero torsion but a non-trivial dilaton field. In the following we will concentrate on solution 1. As we can see, this solution breaks two of the four supersymmetries, related to the positive chirality supersymmetry parameter.

## 2 Fermionic equations of motion

We are now ready to discuss the fermionic equations of motion arising from the ten-dimensional  $N = 1$  supergravity plus super-Yang-Mills Lagrangian which represents the pointlike limit of the heterotic string theory. This Lagrangian has the form [14]:

$$\begin{aligned}
e^{-1}L_{tot} = & e^{-2\Phi} \left\{ -\frac{1}{2}R - \frac{1}{6}H_{MNP}H^{MNP} + 2(\partial_M\Phi)^2 - \frac{1}{4}F_{MN}^\alpha F^{MN\alpha} - \right. \\
& - \frac{1}{2}\bar{\chi}^\alpha \Gamma^M \mathcal{D}_M \chi^\alpha - \frac{1}{2}\bar{\psi}_M \Gamma^{MNP} D_N \psi_P - \psi_M \Gamma^M \partial^N \Phi \psi_N - \bar{\lambda} \Gamma^M D_M \lambda - \\
& - \frac{1}{2}\bar{\psi}_M \Gamma^N \partial_N \Phi \Gamma^M \lambda + \frac{1}{24}H_{RST} \left[ \bar{\psi}_M \Gamma^{MRSTN} \psi_N + 6\bar{\psi}^R \Gamma^S \psi^T + \right. \\
& + \left. 2\bar{\psi}_M (\Gamma^{MRST} - 3g^{MT} \Gamma^{RS}) \lambda \right] - \frac{1}{4}\bar{\chi}^\alpha (\Gamma^{MNP} F_{NP}^\alpha + 2F^M{}_P{}^\alpha \Gamma^P) \psi_M - \left. \right\} \quad (6)
\end{aligned}$$

$$- \frac{1}{4} \bar{\chi}^\alpha \Gamma^{NP} F_{NP}^\alpha \lambda + \frac{1}{24} \text{Tr}(\bar{\chi} \Gamma^{MNP} \chi) H_{MNP} \Big\}.$$

where  $\alpha$  is an  $E_8$  index and where we have redefined the gravitino  $\psi_M$  to make diagonal the kinetic derivative terms of the fermions; our gravitino is related to that of Bergshoeff and de Roo by:

$$\psi_M = \psi_M^{(BdR)} + \frac{\sqrt{2}}{4} \Gamma_M \lambda^{(BdR)}. \quad (7)$$

Moreover to agree with [5], we have made some other field redefinitions:

$$H_{MNP} = \frac{3}{\sqrt{2}} H_{MNP}^{(BdR)}, \quad \lambda = \frac{1}{\sqrt{2}} \lambda^{(BdR)}, \quad \Phi = \frac{3}{2} \log \phi^{(BdR)}. \quad (8)$$

From this Lagrangian it is now easy to derive the fermionic equations of motion, taking into account that all the fermions are represented by Majorana-Weyl spinors: the equation of motion of the gravitino is

$$\begin{aligned} e^{-1} \frac{\delta S}{\delta \psi_M} &= e^{-2\Phi} \left\{ -\Gamma^{MNP} (D_N - \partial_N \Phi) \psi_P + \frac{1}{12} H_{RST} \Gamma^{MRSTN} \psi_N - \right. \\ &\quad - \frac{1}{2} \left( \Gamma^N \partial_N \Phi + \frac{1}{6} H_{RST} \Gamma^{RST} \right) \Gamma^M \lambda - 2\Gamma^M \partial^N \Phi \psi_N \\ &\quad \left. + \frac{1}{2} H^{MNP} \Gamma_N \psi_P - \frac{1}{4} (\Gamma^{MNP} F_{NP}^\alpha - 2F^M{}_P{}^\alpha \Gamma^P) \chi^\alpha \right\} = 0; \end{aligned} \quad (9)$$

the equation for the dilatino is:

$$\begin{aligned} e^{-1} \frac{\delta S}{\delta \bar{\lambda}} &= e^{-2\Phi} \left\{ -2\Gamma^M (D_M - \partial_M \Phi) \lambda + \frac{1}{4} \Gamma^{NP} F_{NP}^\alpha \chi^\alpha - \right. \\ &\quad \left. - \frac{1}{2} \Gamma^M \left( \Gamma^N \partial_N \Phi - \frac{1}{6} \Gamma^{RST} H_{RST} \right) \psi_M \right\} = 0, \end{aligned} \quad (10)$$

and the equation for the gaugino is:

$$\begin{aligned} e^{-1} \frac{\delta S}{\delta \bar{\chi}^\alpha} &= e^{-2\Phi} \left\{ -\Gamma^M (\mathcal{D}_M - \partial_M \Phi) \chi + \frac{1}{12} \Gamma^{MNP} H_{MNP} \chi^\alpha - \right. \\ &\quad \left. - \frac{1}{4} (\Gamma^{MNP} F_{NP}^\alpha + 2F^M{}_P{}^\alpha \Gamma^P) \psi_M - \frac{1}{4} \Gamma^{NP} F_{NP}^\alpha \lambda \right\} = 0. \end{aligned} \quad (11)$$

As we have shown before, supersymmetric solitons are characterised by background fields completely independent of the coordinates of the six-dimensional Minkowskian manifold and depending on some of the remaining coordinates of the four-dimensional curved manifold. To obtain the four-dimensional equations of motion we simply have to neglect the components of the background fields with indices on the six-dimensional flat Minkowski manifold swept out by the five-brane. Supersymmetric solitons half of the supersymmetries, since they annihilate the positive chirality four-dimensional supersymmetry parameter; thus, since we are interested in the possible presence of fermionic zero-modes, we have to study the equations of motion for negative chirality gravitino and gaugino and for positive chirality dilatino. Taking into account that

$$\begin{aligned} (\gamma^{\mu\nu\rho} F_{\nu\rho}^I - 2F_{\rho}^{\mu I} \gamma^{\rho}) \varepsilon_{\pm} &= (\sqrt{g} \varepsilon^{\mu\nu\rho\sigma} \gamma_5 \gamma_{\sigma} F_{\nu\rho}^I - 2F_{\rho}^{\mu I} \gamma^{\rho}) \varepsilon_{\pm} = \\ &= -2(\tilde{F}_{\rho}^{\mu I} \gamma_5 - F_{\rho}^{\mu I}) \gamma^{\rho} \varepsilon_{\pm} = -2F_{\rho}^{\mu I} \gamma^{\rho} (\mathbb{1} + \gamma_5) \varepsilon_{\pm}, \end{aligned} \quad (12)$$

where  $I$  is an  $SU(2)$  index, and  $\mu, \nu, \rho, \sigma$  are curved 4-dimensional indices, we obtain the following equations:

$$-\gamma^{\mu\nu\rho} (D_{\nu} - \partial_{\nu} \Phi) \psi_{\rho}^{-} - 2\gamma^{\mu} \partial^{\nu} \Phi \psi_{\nu}^{-} + \frac{1}{2} H^{\mu\nu\rho} \gamma_{\nu} \psi_{\rho}^{-} = 0, \quad (13)$$

$$-2\gamma^{\mu} (D_{\mu} - \partial_{\mu} \Phi) \lambda_{+} - \gamma^{\mu} \gamma^{\nu} \partial_{\nu} \Phi \psi_{\mu}^{-} + \frac{1}{4} F_{\mu\nu}^I \gamma^{\mu\nu} \chi_{-}^I = 0, \quad (14)$$

$$-\gamma^{\mu} (\mathcal{D}_{\mu} - \partial_{\mu} \Phi) \chi_{-}^I + \frac{1}{12} H_{\mu\nu\rho} \gamma^{\mu\nu\rho} \chi_{-}^I - F_{\rho}^{\mu I} \gamma^{\rho} \psi_{\mu}^{-} = 0. \quad (15)$$

As we can see, the generalised Dirac operator acting on fermions is represented by a triangular matrix, so that its determinant is given by the product of the determinants of the fermionic kinetic operators. We are then authorised to calculate separately the index theorems for each of these fermion operators, without worrying about mixing effects between different fermions[26]. The equations relevant to this calculation are



then:

$$\begin{aligned}
\gamma^{\mu\nu\rho}(D_\nu - \partial_\nu\Phi)\psi_\rho^- + 2\gamma^\mu\partial_\nu\Phi\psi_\nu^- - \frac{1}{2}H^{\mu\nu\rho}\gamma_\nu\psi_\rho^- &= 0, \\
\gamma^\mu(D_\mu - \partial_\mu\Phi)\lambda_+ &= 0, \\
\gamma^\mu(\mathcal{D}_\mu - \partial_\mu\Phi)\chi_-^I - \frac{1}{12}H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\chi_-^I &= 0.
\end{aligned} \tag{16}$$

Defining new dilatino and gaugino fields  $\hat{\lambda} = e^{-\Phi}\lambda$ ,  $\hat{\chi}^I = e^{-\Phi}\chi^I$  we get the simple equations

$$\begin{aligned}
\gamma^\mu D_\mu \hat{\lambda}_+ &= 0, \\
\gamma^\mu \mathcal{D}_\mu \hat{\chi}_-^a - \frac{1}{12}H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\hat{\chi}_-^a &= 0,
\end{aligned} \tag{17}$$

which are well suited to be investigated through the index theorems. Regarding the gravitino equation, we must get rid of the unwanted terms containing the dilaton field; in general this will not be possible. We will however show that for string solitons this is possible.

### 3 Index theorems in presence of torsion

Index theorems represent a fundamental tool in studying the structure of differential elliptic operators like the exterior derivative on forms, the Dirac operators on curved backgrounds or coupled to non-trivial gauge connections, and finally the Rarita-Schwinger operator for the gravitino. They give us information about the existence of zero-modes of these operators and on their number or, better, on certain algebraic sums of these modes. These theorems are well known for manifolds without torsion, that is in the case in which the affine connections  $\Gamma^\lambda_{\mu\nu}$  are symmetric in their lower indices, and thus coincide with the Christoffel connection. In a generic manifold with torsion,

the relation between the affine connection with torsion  $\Gamma^\lambda_{\mu\nu}$  and the spin connection  $\Omega_\mu^{ab}$  is still given by the metricity condition:

$$D_\mu e_\nu^a = \partial_\mu e_\nu^a + \Omega_\mu^a{}_b e_\nu^b - \Gamma^\lambda_{\nu\mu} e_\lambda^a = 0, \quad (18)$$

from which we have

$$\Omega_\mu^{ab} = \omega_\mu^{ab} + e_\nu^a e_\rho^b K^{\nu\rho}{}_\mu, \quad \Gamma^\lambda_{\mu\nu} = \gamma^\lambda_{\mu\nu} + K^\lambda_{\mu\nu}, \quad (19)$$

where  $\gamma^\lambda_{\mu\nu}$  are the Christoffel symbols,  $K^\lambda_{\mu\nu}$  is the contortion tensor

$$K^\lambda_{\mu\nu} = \frac{1}{2}(T_{\mu\nu}{}^\lambda + T^\lambda_{\mu\nu} + T^\lambda_{\nu\mu}), \quad (20)$$

defined in terms of the torsion tensor  $T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}$ , and  $\omega_\mu^{ab}$  is the Levi-Civita spin connection; the commutation relation between the covariant derivatives is:

$$[D_\mu, D_\nu] = \frac{1}{2} R^{ab}{}_{\mu\nu} \Sigma_{ab} + T^\lambda_{\mu\nu} D_\lambda, \quad (21)$$

where  $\Sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$  are the generators of the Lorentz transformations. The derivation of the chiral anomaly in the Riemann-Cartan space, following the articles by Yajima and Kimura [13], is obtained through De Witt's heat kernel method [23] applied to the Dirac operator with a torsionful spin connection. The fundamental step is to introduce a new spin connection which cancels the linear term in the covariant derivative in (21) to calculate all the quantities depending on the spin connection with the new spin connection with torsion. After some manipulations it turns out that the chiral anomaly is given by  $A(x) = \frac{1}{16\pi^2} \text{Tr} \gamma_5 [a_2]$ , where  $[a_2]$  is the so-called second De Witt-Minakshisundaram-Seeley coefficient:

$$\begin{aligned} [a_2] &= \frac{1}{12} Y^{\mu\nu} Y_{\mu\nu} + \frac{1}{180} (R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - R^{\mu\nu} R_{\mu\nu}) + \\ &+ \frac{1}{6} \Box \left( Z - \frac{1}{5} R \right) + \frac{1}{2} \left( Z - \frac{1}{6} R \right)^2. \end{aligned} \quad (22)$$

The tensor  $Y_{\mu\nu}$  is given by

$$Y_{\mu\nu} = \frac{1}{4}\tilde{R}^{ab}{}_{\mu\nu}\gamma_{ab} - F_{\mu\nu}, \quad (23)$$

where the tilde stands for objects calculated with the new spin connection which has three times the original torsion, and, in case of totally antisymmetric torsion,

$$Z = \frac{1}{4}R + \frac{1}{8}S^\mu S_\mu + \frac{1}{4}\gamma_5 \nabla_\mu S^\mu - \frac{1}{2}\gamma^{\mu\nu} F_{\mu\nu}, \quad (24)$$

where  $S_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}K^{\nu\rho\sigma}$ ,  $\varepsilon_{\mu\nu\rho\sigma}$  is a covariantly constant tensorial density and  $\nabla_\mu$  means covariant differentiation with respect to the Levi-Civita connection. Finally, using the reduction formulae of the  $\gamma_{ab}$  matrices Yajima and Kimura obtained:

$$\begin{aligned} A(x) = & \frac{1}{16\pi^2} \left\{ \varepsilon_{abcd} \left( -\frac{\dim V}{48} \tilde{R}^{ab\mu\nu} \tilde{R}^{cd}{}_{\mu\nu} + \frac{1}{2} \text{tr}(F^{ab} F^{cd}) \right) + \right. \\ & \left. + \frac{\dim V}{6} \square(\nabla_\mu S^\mu) + \frac{\dim V}{12} R \nabla_\mu S^\mu + \frac{\dim V}{8} S^\nu S_\nu \nabla_\mu S^\mu \right\} \end{aligned} \quad (25)$$

where  $\dim V$  is the dimension of the representation  $V$  of the gauge group  $G$  the Dirac fermion belongs to (if  $G = SU(2)$ , then  $\dim V = 2t+1$ ). This result is valid for manifold without boundary. In the case of supergravity where the role of torsion is played by the totally antisymmetric field-strength tensor  $H_{\mu\nu\rho}$  which satisfies the Bianchi identities:

$$\varepsilon^{\mu\nu\rho\sigma} \nabla_\mu H_{\nu\rho\sigma} = 0, \quad (26)$$

equivalent to  $\nabla_\mu S^\mu = 0$ , we obtain a powerful simplification which allows us to extend the index theorem to manifolds with boundaries. The volume contribution to the anomaly is now simply:

$$A(x) = \frac{1}{16\pi^2} \varepsilon_{abcd} \left\{ -\frac{\dim V}{48} \tilde{R}^{ab\mu\nu} \tilde{R}^{cd}{}_{\mu\nu} + \frac{1}{2} \text{tr}(F^{ab} F^{cd}) \right\}, \quad (27)$$

which translated in terms of differential forms and characteristic polynomials becomes:

$$\text{ind } \mathcal{D}(M, V) = \frac{2t+1}{192\pi^2} \int_M \text{tr}(\tilde{R}(\tilde{\Omega}) \wedge \tilde{R}(\tilde{\Omega})) - \frac{1}{8\pi^2} \int_M \text{Tr}(F \wedge F), \quad (28)$$

where  $\tilde{\Omega} = \omega - 3K$ ,  $M$  is a four dimensional manifold and where the curvature two-form  $\tilde{R}$  is calculated from the connection with *minus* three times the torsion, due to the exchange symmetry of the torsionful Riemann curvature tensor:

$$\tilde{R}_{\mu\nu\rho\sigma}(\omega + K) = \tilde{R}_{\rho\sigma\mu\nu}(\omega - K), \quad (29)$$

when the torsion satisfies the Bianchi identities [24]. Exploiting the properties of the invariant polynomials, the straightforward generalisation we propose is:

$$\begin{aligned} \text{ind } \mathcal{D}_V(M, \partial M) &= \frac{\dim V}{192\pi^2} \left[ \int_M \text{tr} \tilde{R}(\tilde{\Omega}) \wedge \tilde{R}(\tilde{\Omega}) - \int_{\partial M} \text{tr } \tilde{\theta} \wedge \tilde{R}(\tilde{\Omega}) \right] - \\ &- \frac{1}{8\pi^2} \int_M \text{Tr}_V(F \wedge F) - \frac{1}{2} [\eta_D(\partial M) + h_D(\partial M)] \end{aligned} \quad (30)$$

where the Atiyah-Patodi-Singer (APS)  $\eta$ -invariant [25] is evaluated by solving the three-dimensional Dirac equation with spin connection  $(\omega + K)$  on the boundary of the manifold and the second fundamental form is:

$$\tilde{\theta}^a_b = (\tilde{\Omega})^a_b - (\tilde{\Omega})^a_{(0)b}. \quad (31)$$

If the manifold has more than one boundary, the APS invariant is given by the algebraic sum of the invariants of each boundary, with sign  $+$  or  $-$  depending on the orientation of the boundaries. The “0” index means that  $\omega_0$  should be evaluated on a manifold which admits a product metric on the boundary and that coincides with  $\omega$  in the bulk, while  $K_0$  is simply equal to  $K$  and so the  $K$  contribution disappears from the definition of the second fundamental form.

## 4 Fermionic zero-modes around CHS solitons

Let us now go to our specific case, the CHS one-instanton solution: the instanton gauge field in the language of differential forms reads  $A^I \equiv A^I_\mu dx^\mu = \frac{2\rho^2 \bar{\sigma}^I}{\rho^2 + r^2}$  where the  $\bar{\sigma}^I$  are the  $SU(2)$  right-invariant 1-forms. The standard embedding between the

gauge connection and the spin connection translates into the relation  $\Omega_+^{ab} = \frac{1}{2}\bar{\eta}_{ab}^I A^I$  where we have chosen the orientation  $\varepsilon_{1230} = +1$ . The vierbein is taken to be  $e^a = e^\Phi(dr, r\bar{\sigma}^1, r\bar{\sigma}^2, r\bar{\sigma}^3)$ , while the Levi-Civita spin connection components are:

$$\omega^{i0} = (1 + r\Phi')\bar{\sigma}^i, \quad \omega^{jk} = \varepsilon^{jki}\bar{\sigma}^i, \quad (32)$$

where  $\Phi' = \frac{d\Phi}{dr}$ . In the chosen frame, the 1-form  $H^{ab} = H_\mu^{ab}dx^\mu$  has components  $H^{i0} = 0$  and  $H^{jk} = -r\Phi'\varepsilon^{jki}\bar{\sigma}^i$  and the dilaton field is simply given by  $e^{2\Phi} = e^{2\Phi_0} \left(1 + \frac{\rho^2}{r^2}\right)$ , where we have placed the instanton in the origin so that:

$$\begin{aligned} \Omega_+^{i0} &= (1 + r\Phi')\bar{\sigma}^i, & \Omega_+^{jk} &= (1 - r\Phi')\varepsilon^{jki}\bar{\sigma}^i, \\ \Omega_-^{i0} &= (1 + r\Phi')\bar{\sigma}^i, & \Omega_-^{i0} &= (1 + r\Phi')\varepsilon^{jki}\bar{\sigma}^i. \end{aligned} \quad (33)$$

Since the gauge instanton lives in an  $SU(2)$  subgroup of the  $E_8 \times E_8$  gauge group, we have to decompose the adjoint representation of the latter in terms of the representations of the  $SU(2)$  the instanton lives in. By decomposing the second  $E_8$  with respect to its maximal subgroup  $E_7 \times SU(2)$ , the adjoint **248** of  $E_8$  breaks in

$$\mathbf{248} = (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{56}, \mathbf{2}) \oplus (\mathbf{133}, \mathbf{1}). \quad (34)$$

We are left with calculating the index theorem only for the singlet representation, the fundamental and the adjoint representations of  $SU(2)$ . Let us begin with the gaugino kinetic operator: it contains neither  $\Omega_+$  nor  $\Omega_-$  but, after a simple conformal rescaling, has the form:

$$\gamma^\mu \left( D_\mu - \frac{1}{12} H_{\mu\nu\rho} \gamma^{\nu\rho} \right) \chi^I = 0. \quad (35)$$

The volume contribution to the Dirac index theorem is then:

$$\begin{aligned} \text{ind } \not{D}_V(\text{volume}) &= \frac{2t+1}{192\pi^2} \int_M \text{tr}(\tilde{R}(\omega + H) \wedge \tilde{R}(\omega + H)) - \\ &- \frac{1}{8\pi^2} \int_M \text{Tr}_V(F \wedge F). \end{aligned} \quad (36)$$

In presence of the CHS soliton, the generalised curvature 2-form has components:

$$\begin{aligned}\tilde{R}^{i0} &= (\Phi' + r\Phi'')dr \wedge \bar{\sigma}^i + \varepsilon^i_{jk}\bar{\sigma}^j \wedge \bar{\sigma}^k(r^2\Phi'^2 + r\Phi'), \\ \tilde{R}^{jk} &= (2r^2\Phi'^2 + 2r\Phi')\bar{\sigma}^k \wedge \bar{\sigma}^j - \varepsilon^{jk}_i(\Phi' + r\Phi'')dr \wedge \bar{\sigma}^i.\end{aligned}\quad (37)$$

A straightforward calculation yields:

$$\text{ind } \mathcal{D}_V(\text{volume}) = \frac{2t+1}{12} + \frac{2}{3}t(t+1)(2t+1); \quad (38)$$

as far as the boundary corrections are concerned, there are two contributions from the two boundaries of the single semi-wormhole solution, the  $S^3$  in the origin and the  $S^3$  at infinity; the second fundamental form has only components normal to the boundaries, that is  $\theta^i_0 = \omega^i_0 \neq 0$ . In this case since  $\omega^i_0$  vanishes in the origin and the curvature vanishes at infinity, there is no local boundary correction to the index theorem. Therefore we are left with computing the non-local Atiyah-Patodi-Singer  $\eta$ -invariant correction defined by [25]

$$\lim_{s \rightarrow 0} \eta(\mathcal{D}_V, \partial M, s) = \lim_{s \rightarrow 0} \sum_{\lambda \neq 0} |\lambda|^{-s} \text{sign } \lambda, \quad (39)$$

and the dimension  $h(\mathcal{D}_V, \partial M)$  of the space of the harmonic functions of the operator  $\mathcal{D}_V^2$  on the boundary, that is the number of zero eigenvalues of the Dirac operator on the boundary. The calculation of the  $\eta$ -invariant on the  $S^3$  at infinity is trivial, since the torsion vanishes and from the calculation of Hitchin [27] we get  $\eta(S^3_\infty) = 0$ , while the same calculation is much less trivial in the origin where the torsion contributes. For the Dirac singlet equation one has simply:

$$\mathcal{D}_1 \chi = \bar{\sigma}^\mu \left( \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} - \frac{1}{12} H_\mu^{ab} \gamma_{ab} \right) \chi = 0, \quad (40)$$

and the Dirac operator on the  $S^3$  in the origin becomes:

$$-i \mathcal{D}_1 = 2 \begin{pmatrix} K_3 & K_- \\ K_+ & -K_3 \end{pmatrix} + \mathbb{1}. \quad (41)$$

where the operators  $K_3, K_\pm$  are defined in Appendix A. The details of this calculation also can be found in Appendix A and the final result for the singlet is

$$\eta(\mathcal{D}_\perp, \partial M) = \lim_{s \rightarrow 0} \sum_{k \in \mathbf{N}} \frac{2(l+1)}{(l+1)^s} = 2\zeta(-1, 1) = -\frac{1}{6}; \quad (42)$$

taking into account that the 3-sphere in the origin has orientation opposite to the 3-sphere at infinity which we choose positively oriented, and taking the sum of (38) and (42) we immediately get:

$$\text{ind } \mathcal{D}_\perp(M, \partial M) = \frac{2t+1}{12} \Big|_{t=0} + \frac{1}{2} \cdot -\frac{1}{6} = 0. \quad (43)$$

Thus there are no normalizable zero-modes of the Dirac singlet equation, just as in any manifold without torsion which has a (anti)self-dual curvature two-form, like flat space and gravitational instantons [28]. Let us now study the Dirac index for the other two representations of  $SU(2)$  First of all the gaugino kinetic operator on the boundary has the form:

$$-i\mathcal{D}\chi^I = -i\bar{\sigma}^i \left[ \left( \partial_i + \frac{1}{4}\omega_i^{ab}\gamma_{ab} - \frac{1}{12}H_i^{ab}\gamma_{ab} \right) \delta_K^I + \frac{1}{2}if^I_{JK}A_i^J \right] \chi^K = 0, \quad (44)$$

where  $i, a, b = 1, 2, 3$  and where  $f^I_{JK}$  are the structure constants of the gauge group. First we calculate the APS invariant for the fundamental representation of  $SU(2)$ : since  $\lim_{r \rightarrow 0} A^I = 2\bar{\sigma}^I$ , after some manipulations we get [15, 29]

$$-i\mathcal{D}_2|_{\partial M} = \begin{pmatrix} -i\mathcal{D}_\perp & 0 \\ 0 & -i\mathcal{D}_\perp \end{pmatrix} + \begin{pmatrix} \tau_3 & \tau_- \\ \tau_+ & -\tau_3 \end{pmatrix}. \quad (45)$$

where  $\tau_i$  are the Pauli matrices and  $\tau_\pm = \tau_1 \pm i\tau_2$ . The calculations are shown in Appendix B and it turns out that

$$\eta(\mathcal{D}_2, \partial M) = -\frac{1}{3}; \quad (46)$$

summing the contributions of (46) and (38) we get:

$$\text{ind}(\mathcal{D}_2, M, \partial M) = \frac{2t+1}{12} + \frac{2}{3}t(t+1)(2t+1) + \frac{1}{2}\eta(\mathcal{D}_2, \partial M) =$$

$$= \frac{1}{6} + 1 - \frac{1}{6} = 1, \quad (47)$$

that is we have only one zero-mode of the Dirac operator belonging to the fundamental representation of  $SU(2)$ , the same result than in the flat space 't Hooft instanton case, arising here from non-trivial cancellations between the gravitational term and the non-local boundary corrections. As for the triplet case, we follow the same steps and we finally find that the Dirac index for the  $SU(2)$  triplet becomes:

$$\text{ind } \mathcal{D}_{\underline{3}}(M, \partial M) = \frac{1}{4} + 4 - \frac{1}{4} = 4. \quad (48)$$

The calculations for this case can be found in Appendix C. As in the previous case, we find a non trivial cancellation between the gravitational part and the  $\eta$ -correction which leaves us with the same result as for flat space.

If, before making the conformal rescaling on the gaugino, we had used explicitly the form of  $H_{\mu\nu\rho}$ , we would have found that the torsion contribution is comparable to the rescaled term so that it could be rescaled away together with all the other terms containing the dilaton. In this case we should apply to the gaugino the usual index theorem without torsion. As a simple exercise let us verify that our previous results are consistent with this picture. The curvature 2-form now has the form:

$$R^i_0 = (\Phi' + r\Phi'')dr \wedge \bar{\sigma}^i, \quad R^j_k = (r^2\Phi'^2 + 2r\Phi')\bar{\sigma}^j \wedge \bar{\sigma}^i, \quad (49)$$

and  $\int_M \text{tr}(R(\omega) \wedge R(\omega)) = 0$ , so that the bulk contribution to the index theorem comes entirely from the gauge fields; as for the  $\eta$ -invariant, it must be calculated on two  $S^3$  and for both of them it is zero. The indices for Dirac fermions are then:

$$\text{ind } \mathcal{D}_{\underline{1}}(M, \partial M) = 0, \quad \text{ind } \mathcal{D}_{\underline{2}}(M, \partial M) = 1, \quad \text{ind } \mathcal{D}_{\underline{3}}(M, \partial M) = 4, \quad (50)$$

exactly the result we obtained through a separate treatment of the torsion term in the equations of motion.



As for the dilatino, the calculation is trivial, since neither torsion nor gauge fields couple to its kinetic operator. Using our previous results, we obtain immediately that

$$\text{ind}(\not{D}_\lambda, M, \partial M) = 0; \quad (51)$$

in principle one should be careful in making singular rescalings of the fermionic fields to get rid of the dilatonic terms in the kinetic operators and should wonder whether the number of zero-modes of the rescaled fermions is the same of the original ones. As a check we have verified that the original dilatino kinetic operator does not admit zero-modes; using the fact that possible dilatino zero-modes should have the form

$$\delta\lambda_+ = -\frac{1}{4} \left( \gamma^\mu \partial_\mu \Phi - \frac{1}{6} H_{\mu\nu\rho} \gamma^{\mu\nu\rho} \right) \varepsilon_- = -\frac{1}{2} \gamma^\mu \partial_\mu \Phi \varepsilon_-, \quad (52)$$

and putting this field configuration in the equation of motion, if we make the plausible Ansatz  $\varepsilon_- = e^{p\Phi} \eta$  with constant  $\eta$ , we find a non-normalizable solution for  $p = \frac{5}{2}$ . We therefore find no zero-modes for the dilatino qualitatively justifying the singular conformal rescaling we applied before. Regarding the gravitino field, using the explicit form of the metric and computing the spin connection appearing in the covariant derivative, we obtain an alternative form of the equation of motion for the gravitino:

$$\gamma^{\mu\nu\rho} \hat{D}_\nu \psi_\rho^- + 2\partial^\mu \Phi \gamma^\nu \psi_\nu^- - \frac{1}{2} H^{\mu\nu\rho} \gamma_{\nu\rho} \psi^- = 0, \quad (53)$$

where the covariant derivative  $\hat{D}_\mu$  is evaluated with respect to the metric  $\hat{g}_{\mu\nu} = e^{-2\Phi} \delta_{\mu\nu}$ . Now we can make a gauge choice on the gravitino which does not affect the physics of the problem and we choose a condition of  $\gamma$ -tracelessness; with this information we can immediately throw away the second term in the equation of motion and we are left with just a torsion term. Working out the index theorem for the Rarita-Schwinger operator with torsion has been proved so far very hard to complete and we are still working on it; moreover the coupling of  $H_{\mu\nu\rho}$  to the gravitino does not seem to be compatible with its interpretation as the antisymmetric part of the affine connection

so that it is not clear how to generalise the index theorem to cope with this specific case. We would however like to show that if we use the explicit form of the torsion in this solution, we can absorb the torsion term through a Weyl rescaling of the gravitino field: in fact,

$$H^{\mu\nu\rho}\gamma_\nu\psi_\rho^- = -\sqrt{G}\varepsilon^{\mu\nu\rho\sigma}\partial_\sigma\Phi\gamma_\nu\psi_\rho^- = \Gamma^{\mu\nu\rho}\partial_\nu\Phi\psi_\rho^-, \quad (54)$$

and defining a new gravitino  $\hat{\psi}_\mu = e^{-\Phi/2}\psi_\mu$  we obtain a new equation of motion:

$$\gamma^{\mu\nu\rho}\hat{D}_\nu\hat{\psi}_\rho^- = 0, \quad (55)$$

and we can work with the usual index theorem. It remains to verify whether this procedure is correct, through an extension of the Rarita-Schwinger index formula to the case of coupling to torsion. In absence of torsion, the index of the Rarita-Schwinger operator is given by the formula:

$$\begin{aligned} \text{ind}D_{RS}(M, \partial M) &= -\frac{21}{192\pi^2} \left[ \int_M \text{tr} R(\omega) \wedge R(\omega) - \int_{\partial M} \text{tr} \theta(\omega) \wedge R(\omega) \right] \\ &\quad - \frac{1}{2}[\eta_{RS}(\partial M) + h_{RS}(\partial M)]. \end{aligned} \quad (56)$$

In this case we must be careful in computing curvature 2-forms and fundamental forms with the metric  $\hat{g}_{\mu\nu}$ . One can check that Rarita-Schwinger  $\eta$ -invariant on  $S^3$  is zero so that there are no normalizable solution to the Rarita-Schwinger equation. We have also explicitly checked that, by making the rather general ansatz  $\varepsilon_- = e^{p\Phi}\eta$  with  $p$  and  $\eta$  constants, the original Rarita-Schwinger operator has no zero-modes, enforcing our qualitative results. From the property of factorisation of the generalised Dirac determinant, we can infer that the fermionic zero-modes could in principle have three different forms:

$$\begin{pmatrix} \hat{\psi}_\mu^{(0)} \\ \hat{\chi}^{a'} \\ \hat{\lambda}' \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \hat{\chi}_{(0)}^a \\ \hat{\lambda}'' \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ \hat{\lambda}^{(0)} \end{pmatrix}, \quad (57)$$

where the index “0” stands for the zero-modes, while the other entries are relative to the solution of the fermionic equations of motion in presence of zero-modes; from our analysis we have found that only solutions of the second kind can exist and that there is no space for gravitino condensates. The four gaugino triplet zero-modes are expected to correspond to the breaking of two supersymmetries and of two superconformal symmetries of the theory as it happens in the usual flat Super-Yang-Mills case and to have the form

$$\hat{\chi}^{(0)} = -\frac{1}{4}e^{-\Phi}F_{\mu\nu}\gamma^{\mu\nu}\eta, \quad (58)$$

where  $\eta = \eta_0 + \rho^{-1}x^\mu\gamma_\mu\bar{\varepsilon}_0$  and  $\eta_0, \bar{\varepsilon}_0$  are constant spinors. As for the dilatino and the gravitino, from a more geometrical point of view we would not expect to find any zero-modes, since zero-modes, at least in the single instanton case, are related to broken symmetries and in this case they are just supersymmetry and superconformal symmetry leaving no space for other zero-modes to exist. We have also found explicitly that associated to the two “supersymmetric” gaugino zero-modes the dilatino has a non trivial normalizable configuration  $\hat{\lambda}_+ = -\frac{1}{2}\gamma^\mu\partial_\mu\Phi\eta_0$  which satisfies the equations of motion for the dilatino in presence of the gaugino zero-modes, while for the other two zero-modes the dilatino field is  $\hat{\lambda}_+ = -\frac{1}{2}\gamma^\mu\partial_\mu\Phi(\rho^{-1}x^\lambda\gamma_\lambda\bar{\varepsilon}_0)$  which is not normalizable.

## 5 Conclusions

We have set up a machinery which allows the calculation of the Dirac index theorem both in presence and in absence of torsion; its usefulness stems out from the fact that torsion appears naturally in all superstring inspired supergravity theories and it couples to fermions. It remains to examine the gravitino case which does not couple to torsion according to naive expectations; it has been proved that the Rarita-Schwinger index is not affected by the presence of torsion [30], but unfortunately the kind of coupling

taken into account, although consistent, is not the one that appears in supergravity Lagrangians. At this point it would be interesting to apply the above method to the computation of topological invariants for string solutions related by Buscher's duality [32]. If T-duality is sensibly performed, it relates models with torsion and conformally flat metrics to models without torsion and non-trivial metrics; it has been shown, for example, that the dual of CHS is a kind of black holes [31]. In general, however, if the Killing vector under which we perform the duality has some zeros the dual solution is singular so that the computation may require a more accurate analysis. When a conformal field theory approach is applicable one may find a (non-local) correspondence among vertex operators so that the number of zero modes should agree between T-dual solutions. Anyway our method has a wider applicability being independent of the possibility of a more formal CFT approach and could provide a new insight on the behaviour of all topological invariants under these duality transformations. We have restricted our attention to  $D = 4$ , but it is straightforward in principle to extend the analysis to higher dimensional manifolds with torsion. In the compact case this has been done by Rohm and Witten [11]; recently however non-compact manifolds seem to receive new interest [33].

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# Appendix A

Let us define the right-invariant one-forms:

$$\bar{\sigma}_i = \frac{1}{r^2} \bar{\eta}_{i\mu\nu} x^\mu dx^\nu, \quad (59)$$

or in terms of Euler angles:

$$\begin{aligned} \bar{\sigma}_x &= \frac{1}{2}(\sin \phi \, d\theta - \cos \phi \sin \theta \, d\psi), \\ \bar{\sigma}_y &= -\frac{1}{2}(\cos \phi \, d\theta + \sin \phi \sin \theta \, d\psi), \\ \bar{\sigma}_z &= \frac{1}{2}(d\phi + \cos \theta \, d\psi). \end{aligned} \quad (60)$$

Their dual vector fields are

$$K_i = -\frac{i}{2} \bar{\eta}_i^{\mu\nu} x_\mu \partial_\nu, \quad (61)$$

which satisfy the  $SU(2)$  algebra:  $[K_i, K_j] = -i\varepsilon_{ij}^k K_k$ . The Dirac singlet equation on a generic boundary is simply:

$$\not{D}_\perp \chi = \bar{\sigma}^\mu \left( \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} - \frac{1}{12} H_\mu^{ab} \gamma_{ab} \right) \chi = 0, \quad (62)$$

and the Dirac operator on the  $S^3$  in the origin becomes:

$$-i \not{D}_\perp = 2 \begin{pmatrix} K_3 & K_- \\ K_+ & -K_3 \end{pmatrix} + \mathbb{1}, \quad (63)$$

where  $K_\pm = K_1 \pm iK_2$ . To calculate the eigenvalues of this operator let us act with it upon a state  $\Psi$  whose entries are the  $SU(2)$  rotation matrices  $D_{n,m}^l(\theta, \phi, \psi)$

$$\Psi = \begin{pmatrix} D_{n,m-1}^l \\ D_{n,m}^l \end{pmatrix}. \quad (64)$$

The Dirac operator becomes:

$$-i \not{D}_\perp \Psi = 2 \begin{pmatrix} m - \frac{1}{2} & a \\ a & -m + \frac{1}{2} \end{pmatrix} \Psi, \quad (65)$$

where  $a = \sqrt{(l+m)(l-m+1)}$ . The eigenvalues of this matrix are  $2l+1, -2l-1$  with multiplicities  $d = 2l(2l+1)$  since we restrict  $m$  to the range  $-l+1 \leq m \leq l$ . Otherwise some components of the eigenvector would lose their meaning and we have to study these limiting cases separately [29]:

1.  $m = -l$ . In this case the first component of the eigenvector must be set to zero and we have a single eigenvector with eigenvalue  $2l+1$  and multiplicity  $d = 2l+1$  equal to the degeneracy of the quantum number  $n$ ;
2.  $m = l+1$ . The last component of the eigenvector must be set to zero and again we have one eigenvector with eigenvalue  $2l+1$  and multiplicity  $d = 2l+1$ .

The contribution to the  $\eta$ -invariant from the general cases is identically zero, so we have just to evaluate the contribution coming from the limiting cases:

$$\eta(\mathcal{D}_\perp, \partial M) = \lim_{s \rightarrow 0} \sum_{l=0}^{\infty} \frac{2(2l+1)}{(2l+1)^s}, \quad (66)$$

where the sum runs over integers and half-integers. Turning to a sum over integers we can reexpress this sum in term of generalised Riemann  $\zeta$ -functions:

$$\eta(\mathcal{D}_\perp, \partial M) = 2\zeta(-1, 1) = -\frac{1}{6}. \quad (67)$$

## Appendix B

In this appendix we will explicitly perform the calculation of the  $\eta$ -invariant for a gaugino in the fundamental representation of  $SU(2)$ . The Dirac equation on the boundary is

$$-i\mathcal{D}_2|_{\partial M} = \begin{pmatrix} -i\mathcal{D}_\perp & 0 \\ 0 & -i\mathcal{D}_\perp \end{pmatrix} + \begin{pmatrix} \tau_3 & \tau_- \\ \tau_+ & -\tau_3 \end{pmatrix}. \quad (68)$$

We now have to look for the eigenvalue spectrum of the matrix  $\mathcal{M}$  defined by

$$\mathcal{M} = 2 \begin{pmatrix} K_3 + 1 & K_- & 0 & 0 \\ K_+ & -K_3 & 1 & 0 \\ 0 & 1 & K_3 & K_- \\ 0 & 0 & K_+ & -K_3 + 1 \end{pmatrix} + \mathbb{1}. \quad (69)$$

The form of this matrix suggests us that we can solve the eigenvalue equation  $\mathcal{M}\Psi = \lambda\Psi$  by expanding  $\Psi$  in terms of the  $SU(2)$  rotation matrices  $D_{n,m}^l(\theta, \phi, \psi)$  which are defined in the appendix. We are left with a finite dimensional eigenvalue problem for a  $4 \times 4$  matrix; for generic, given values of  $l, n, m$  (the operator  $\mathcal{M}$  does not act over the index  $n$ ) there are four eigenvectors that may be written as:

$$\Psi_{n,m}^{(i)l} = \begin{pmatrix} c_1^i D_{n,m-1}^l \\ c_2^i D_{n,m}^l \\ c_3^i D_{n,m}^l \\ c_4^i D_{n,m+1}^l \end{pmatrix}, \quad i = 1, 2, 3, 4. \quad (70)$$

The explicit expression of the coefficients  $c_k^{(i)}$  is irrelevant to our problem. Since the allowed values of  $n, m$  in the (70) are restricted to be  $|m| \leq l - 1$  and  $|n| \leq l$  if  $l \neq 0$ , let us again study the limiting cases separately. The four eigenvalues corresponding to the eigenvectors (70) are

$$\lambda_{n,m}^{(1)l} = 2l + 2, \quad \lambda_{n,m}^{(2)l} = 2l, \quad (71)$$

$$\lambda_{n,m}^{(3)l} = -2l, \quad \lambda_{n,m}^{(4)l} = -2l - 2; \quad (72)$$

they have the same multiplicity  $d$ , given by the product of the degeneracies of the  $n$  and  $m$  quantum numbers:  $d = d_m d_n = (2l + 1)(2l - 1)$ . Since these eigenvalues are symmetrically distributed around zero, they do not contribute to the  $\eta$ -invariant. Let us now investigate the contribution of the limiting cases in which either  $l = 0$  or  $m$  violates the condition  $|m| \leq l - 1$ . As far as the special case  $l = 0$  is concerned, the

matrix  $\mathcal{M}$  becomes simply:

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \mathbb{1}, \quad (73)$$

whose eigenvectors are constant and eigenvalues 2,-2 with multiplicities  $d = 3, 1$ ; the contribution from this case is very easily calculated to be

$$\lim_{s \rightarrow 0} \eta(\mathcal{P}_2, \partial M, s)_{l=0} = \lim_{s \rightarrow 0} \left( \frac{3}{2^s} - \frac{1}{2^s} \right) = 2. \quad (74)$$

Finally let us turn to the other limiting cases; they are dealt with by setting to zero the components of (70) that lose meaning when  $m$  takes its limiting values  $m = \pm l$  or  $m = \pm(l+1)$ . The cases are the following:

1.  $m = l + 1$ . Only the first component in (70) is non-zero; there is one eigenvector with eigenvalue  $2l + 2$  and multiplicity  $d = 2l + 1$ ;
2.  $m = -l - 1$ . The last component in (70) is non-zero: there is again one eigenvector with eigenvalue  $2l + 2$  and multiplicity  $d = 2l + 1$ ;
3.  $m = l$  The first three components in (70) are non vanishing; there are 3 eigenvectors  $\Psi_{l,n}^{(k)l}$ ,  $k = 1, 2, 3$  with eigenvalues  $2l + 2, 2l, -2l - 2$  with multiplicity  $d = 2l + 1$ ;
4.  $m = -l$  The last three components in (70) are non vanishing; there are 3 eigenvectors  $\Psi_{-l,n}^{(k)l}$ ,  $k = 1, 2, 3$  with eigenvalues  $2l + 2, 2l, -2l - 2$  with multiplicity  $d = 2l + 1$ .

The contribution from these four cases is then

$$\begin{aligned} \lim_{s \rightarrow 0} \eta(\mathcal{P}_2, \partial M, s)_{l \neq 0} &= \sum_{l=1/2}^{\infty} \left( \frac{4(2l+1)}{(2l+2)^s} + \frac{2(2l+1)}{(2l)^s} - \frac{2(2l+1)}{(2l+2)^s} \right) = \\ &= 2\zeta(-1, 3) - 2\zeta(0, 3) + 2\zeta(-1, 1) + 2\zeta(0, 1) = -\frac{7}{3}; \end{aligned} \quad (75)$$



adding (75) to (74) and since there are no zero eigenvalues of  $\mathcal{D}_2$  on the boundary we obtain that

$$\eta(\mathcal{D}_2, \partial M) = \eta(\mathcal{D}_2, \partial M)_{l=0} + \eta(\mathcal{D}_2, \partial M)_{l \neq 0} = 2 - \frac{7}{3} = -\frac{1}{3}. \quad (76)$$

## Appendix C

In this appendix we want to calculate explicitly the value of the  $\eta$ -invariant for the adjoint representation of  $SU(2)$ : following the same steps as in appendix B and using for simplicity a form for the generators in the adjoint representation which has  $T_3$  diagonal, that is:

$$\begin{aligned} T_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & 0 \\ -i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \\ T_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \end{aligned} \quad (77)$$

the Dirac operator will take the form:

$$-i\mathcal{D}_3|_{\partial M} = \begin{pmatrix} -i\mathcal{D}_1 & 0 & 0 \\ 0 & -i\mathcal{D}_1 & 0 \\ 0 & 0 & -i\mathcal{D}_1 \end{pmatrix} + \begin{pmatrix} 2\tau_3 & -\sqrt{2}\tau_- & 0 \\ -\sqrt{2}\tau_+ & 0 & \sqrt{2}\tau_- \\ 0 & \sqrt{2}\tau_+ & -2\tau_3 \end{pmatrix}. \quad (78)$$

Again, for generic, given, values of  $l$ ,  $m$  and  $n$ , with  $l \neq 0$ ,  $|n| \leq l$  and  $-l+1 \leq m \leq l-2$  there are six different eigenvectors whose form is given by:

$$\Psi_{nm}^{(i)l} = \begin{pmatrix} c_1^i D_{n,m-1}^l \\ c_2^i D_{n,m}^l \\ c_3^i D_{n,m}^l \\ c_4^i D_{n,m+1}^l \\ c_5^i D_{n,m+1}^l \\ c_6^i D_{n,m+2}^l \end{pmatrix}, \quad i = 1, \dots, 6. \quad (79)$$

The corresponding eigenvalues are:

$$\lambda_{n,m}^{(1)l} = 2l + 3, \quad \lambda_{n,m}^{(2)l} = 2l + 1, \quad (80)$$

$$\lambda_{n,m}^{(3)l} = 2l - 1, \quad \lambda_{n,m}^{(4)l} = -2l - 3, \quad (81)$$

$$\lambda_{n,m}^{(5)l} = -2l - 1, \quad \lambda_{n,m}^{(6)l} = -2l + 1; \quad (82)$$

since they are symmetrically distributed around zero and have the same multiplicity, again they do not contribute to the  $\eta$ -invariant. Let us study the remaining seven limiting cases:

1.  $l = m = n = 0$ . In this case the operator has two constant eigenvectors with eigenvalues 3, -3 and multiplicities  $d=4, 2$  respectively. Their contribution to the  $\eta$  is simply  $\eta_{l=0} = 2$ ;
2.  $m = l + 1$ . Only the first component in (79) is non-zero; there is one eigenvector with eigenvalue  $2l + 3$  and multiplicity  $d = 2l + 1$ ;
3.  $m = -l - 2$ . The last component in (79) is non-zero; we have one eigenvector with eigenvalue  $2l + 3$  and multiplicity  $d = 2l + 1$ ;
4.  $m = l$ . The last three components in (79) must be set to zero; there are three eigenvectors  $\Psi_{nl}^{(k)l}$ , with  $k = 1, 2, 3$  whose eigenvalues are  $2l + 3, 2l + 1, -2l - 3$ , with multiplicity  $d = 2l + 1$ ;
5.  $m = -l - 1$ . This case is analogous to the previous one, except that the first three components in (79) are zero. Three eigenvectors with eigenvalues  $2l + 3, 2l + 1, -2l - 3$ , and multiplicity  $d = 2l + 1$ .
6.  $m = l - 1$  and  $m = -l$ . In both cases we have five eigenvectors with eigenvalues  $2l + 3, 2l + 1, 2l - 1, -2l + 1, -2l - 3$ , with multiplicity  $d = 2l + 1$ .

Putting all these contributions together we obtain:

$$\eta(\mathcal{D}_3, \partial M) = \eta_{l=0} + \eta_{limit} = 2 - \frac{5}{2} = -\frac{1}{2}, \quad (83)$$

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